

# FINITELY GENERATED MODULES OVER QUASI-EUCLIDEAN RINGS

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**ABSTRACT.** Let  $R$  be a unital commutative ring and let  $M$  be an  $R$ -module that is generated by  $k$  elements but not less. Let  $E_n(R)$  be the subgroup of  $GL_n(R)$  generated by the elementary matrices. In this paper we study the action of  $E_n(R)$  by matrix multiplication on the set  $Um_n(M)$  of unimodular rows of  $M$  of length  $n \geq k$ . Assuming  $R$  is moreover Noetherian and quasi-Euclidean, e.g.,  $R$  is a direct sum of finitely many Euclidean rings, we show that this action is transitive if  $n > k$ . We also prove that  $Um_k(M)/E_k(R)$  is equipotent with the unit group of  $R/\mathfrak{a}_1$  where  $\mathfrak{a}_1$  is the first invariant factor of  $M$ . These results encompass the well-known classification of Nielsen non-equivalent generating tuples in finitely generated Abelian groups.

## 1. INTRODUCTION

In this paper rings are supposed unital and commutative. The unit group of a ring  $R$  is denoted by  $R^\times$ . Let  $M$  be a finitely generated  $R$ -module. We denote by  $\text{rk}_R(M)$  the minimal number of generators of  $M$ . For  $n \geq \text{rk}_R(M)$ , we denote by  $Um_n(M)$  the set of *unimodular rows* of  $M$  of length  $n$ , i.e., the set of elements in  $M^n$  whose components generate  $M$ . We consider the action of  $GL_n(R)$  on  $Um_n(M)$  by matrix right-multiplication. Let  $E_n(R)$  be the subgroup of  $GL_n(R)$  generated by the *elementary matrices*, i.e., the matrices which differ from the identity by a single off-diagonal element. Two unimodular rows  $\mathbf{m}, \mathbf{m}' \in Um_n(M)$  are said to be  $E_n(R)$ -*equivalent* if there exists  $E \in E_n(R)$  such that  $\mathbf{m}' = \mathbf{m}E$ . Our chief concern is the description of the orbit set  $Um_n(M)/E_n(R)$  when  $R$  enjoys a cancellation property shared by Euclidean rings, namely: for every  $n \geq 2$  and every  $\mathbf{r} = (r_1, \dots, r_n) \in R^n$ ,

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there exist  $E \in E_n(R)$  and  $d \in R$  such that

$$(1) \quad (d, 0, \dots, 0) = \mathbf{r}E.$$

Rings with the above property are known as quasi-Euclidean rings in the sense of O'Meara and Cooke [O'M65, Coo76] (see [AJLL14, Theorem 11] for equivalence of definitions). A Noetherian quasi-Euclidean ring  $R$  is therefore a *principal ideal ring* (PIR), i.e., a ring whose ideals are principal. It is moreover an elementary divisor ring (see definition below) so that every finitely generated  $R$ -module admits an *invariant factor decomposition*, that is a decomposition of the form

$$(2) \quad R/\mathfrak{a}_1 \times R/\mathfrak{a}_2 \times \dots \times R/\mathfrak{a}_k \text{ with } R \neq \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \dots \supset \mathfrak{a}_k$$

where  $k$  is necessarily equal to  $\text{rk}_R(M)$ . If  $R$  is both Noetherian and quasi-Euclidean, properties (1) and (2) make the study of the action of  $E_n(R)$  on  $\text{Um}_n(R)$  particularly amenable. Indeed, this assumption implies that  $E_n(R)$  acts transitively on  $\text{Um}_n(M)$  for every  $n > \text{rk}_R(M)$  and enables us to exhibit a complete invariant of  $E_n(R)$ -equivalence when  $n = \text{rk}_R(M)$ . These two claims are corollaries of Theorem A below.

**Theorem A.** *Let  $R$  be a Noetherian quasi-Euclidean ring and let  $M$  be a finitely generated  $R$ -module. Let  $R/\mathfrak{a}_1 \times \dots \times R/\mathfrak{a}_k$  be the invariant factor decomposition of  $M$ . Then every row in  $\text{Um}_n(M)$  with  $n \geq k$  is  $E_n(R)$ -equivalent to a row of the form  $(\delta e_1, e_2, \dots, e_k, 0, \dots, 0)$  with  $\delta \in (R/\mathfrak{a}_1)^\times$  and where  $e_i \in M$  is defined by  $(e_i)_i = 1 \in R/\mathfrak{a}_i$  and  $(e_i)_j = 0$  for  $j \neq i$ . If  $n > k$ , then  $\delta$  can be replaced by the identity.*

**Corollary B.** *Let  $M$  be as in Theorem A. The action of  $E_n(R)$  on  $\text{Um}_n(M)$  by matrix multiplication is transitive for every  $n > \text{rk}_R(M)$ .*

**Corollary C.** *Let  $R$  be a Noetherian quasi-Euclidean ring. Let  $\mathfrak{b}_1, \dots, \mathfrak{b}_k$  be proper ideals of  $R$  and set  $\mathfrak{b} = \mathfrak{b}_1 + \dots + \mathfrak{b}_k$ . Denote by  $M$  the  $R$ -module  $R/\mathfrak{b}_1 \times \dots \times R/\mathfrak{b}_k$  and for  $\mathbf{m} = (m_i) \in M^k$  denote by  $\det(\mathbf{m})$  the determinant of the matrix whose coefficients are the images in  $R/\mathfrak{b}$  of the  $(m_i)_j$ 's via the natural maps  $R/\mathfrak{b}_j \rightarrow R/\mathfrak{b}$ . Then  $\mathbf{m}, \mathbf{m}' \in \text{Um}_k(M)$  are  $E_k(R)$ -equivalent if and only if  $\det(\mathbf{m}) = \det(\mathbf{m}')$ .*

Suppose  $k = \text{rk}_R(M)$ . Then the ideal  $\mathfrak{b}$  of Corollary C coincides with the first invariant factor  $\mathfrak{a}_1$  of  $M$  (cf. proof). Putting Theorem A and Corollary C together, we see that  $\mathbf{m} \in \text{Um}_k(M)$  is  $E_k(R)$ -equivalent to  $(\det(\mathbf{m})e_1, e_2, \dots, e_k)$  where  $(e_i)$  is as in Theorem A and  $\det$  as in Corollary C. Therefore the map  $\mathbf{g} \mapsto \det(\mathbf{g})$  induces a bijection from  $\text{Um}_k(M)/E_k(R)$  onto  $(R/\mathfrak{a}_1)^\times$ . The latter bijection endows  $\text{Um}_k(M)/E_k(R)$  with an Abelian group structure. The subject whether  $\text{Um}_k(R)/E_k(R)$  has a group structure for  $R$  a commutative ring

of finite Krull dimension is well studied [VS76, vdK83, vdK89, Rao98, Fas11] but we don't know of any similar results for modules. By analogy with [MR87, Definition 11.3.9], we can define the elementary rank of a finitely generated  $R$ -module  $M$ , for any associative ring  $R$  with identity, as the least integer  $e$  such that the action of  $E_n(R)$  on  $\text{Um}_n(M)$  is transitive for all  $n > e$ . This rank is not less than  $\text{rk}_R(M) - 1$  and not greater than  $\text{rk}_R(M) - 1 + \text{sr}(M)$ , where  $\text{sr}(M)$  is the stable rank of  $M$ , a natural generalization of the Bass stable rank of rings to modules [MR87, Definition 6.7.2]. We showed in our situation that the elementary rank is  $\text{rk}_R(M) - 1$  if  $R/\mathfrak{a}_1$  has 2 elements and coincides with  $\text{rk}_R(M)$  otherwise.

Specifying the above results to  $R = \mathbb{Z}$  yields the characterization of Nielsen equivalent generating tuples in finitely generated Abelian groups. This characterization was obtained in part by several authors [NN51, LM93, DG99] and reaches its complete form in [Oan11]. In order to present it, we introduce the following definitions. Given a finitely generated group  $G$ , denote by  $\text{rk}(G)$  the minimal number of generators of  $G$ . For  $n \geq \text{rk}(G)$ , let  $V_n(G)$  be the set of *generating  $n$ -vectors* of  $G$ , i.e., the set of elements in  $G^n$  whose components generate  $G$ . Two generating  $n$ -vectors are said to be *Nielsen equivalent* if they can be related by a finite sequence of transformations of  $G^n$  taken in the set  $\{L_{ij}, I_i; 1 \leq i \neq j \leq n\}$  where  $L_{ij}$  and  $I_i$  replace the component  $g_i$  of  $\mathbf{g} = (g_1, \dots, g_n) \in G^n$  by  $g_j g_i$  and  $g_i^{-1}$  respectively and leave the other components unchanged.

**Corollary D.** *Let  $G$  be a finitely generated Abelian group whose invariant factor decomposition is*

$$\mathbb{Z}_{d_1} \times \cdots \times \mathbb{Z}_{d_k}$$

*with  $1 \neq d_1 \mid d_2 \mid \cdots \mid d_k$ ,  $d_i \geq 0$  and where  $\mathbb{Z}_{d_i}$  stands for  $\mathbb{Z}/d_i\mathbb{Z}$  (in particular  $\mathbb{Z}_0 = \mathbb{Z}$ ). Then every generating  $n$ -vector  $\mathbf{g}$  with  $n \geq k$  is Nielsen equivalent to  $(\delta e_1, e_2, \dots, e_k, 0, \dots, 0)$  for some  $\delta \in (\mathbb{Z}_{d_1})^\times$  and where  $e_i \in G$  is defined by  $(e_i)_i = 1 \in \mathbb{Z}_{d_i}$  and  $(e_i)_j = 0$  for  $j \neq i$ .*

- *If  $n > k$ , then  $\delta$  above can be replaced by the identity.*
- *If  $n = k$  then  $\delta$  must be  $\pm \det(\mathbf{g})$  with  $\det$  defined as in Corollary C.*

*In particular  $G$  has only one Nielsen equivalence class of generating  $n$ -vectors for  $n > k$  while it has  $\max(\varphi(d_1)/2, 1)$  Nielsen equivalence classes of generating  $k$ -vectors where  $\varphi$  denotes the Euler totient function extended by  $\varphi(0) = 0$ .*

Our results allow further applications to the study of Nielsen equivalence in split extensions of Abelian groups by cyclic or free Abelian groups. Consider for instance an infinite cyclic group  $C$  and denote by  $\mathbb{Z}[C]$  its integral group ring. Let  $R$  be a quasi-Euclidean quotient of  $\mathbb{Z}[C]$ , e.g.,  $R = \mathbb{Z}_n[C]$  for  $n$  a square-free integer, and let  $M$  be a finitely generated  $R$ -module. Then the image of  $C$  in

$R$  is a subgroup of  $R^\times$  so that  $C$  acts naturally on  $M$  by automorphisms. Let  $G = M \rtimes C$  be the corresponding semi-direct product. Let  $T$  be the subgroup of  $R^\times$  generated by the images of  $-1$  and  $C$ , set  $\Lambda = R/\mathfrak{a}_1$  where  $\mathfrak{a}_1$  is the first invariant factor of  $M$ , and let  $T_\Lambda$  be the image of  $T$  in  $\Lambda^\times$ . Our results allow us to show that the set of Nielsen equivalence classes of generating  $k$ -tuples of  $G$  is equipotent with  $\Lambda^\times/T_\Lambda$  for  $k = \text{rk}(G)$  [Guy16b]. (See also [Guy16a, Corollary 4.ii] for an application of Corollary B).

We introduce now definitions and preliminary results that will be used by the proofs of our statements in Section 2.

A ring  $R$  is said to be an *elementary divisor ring* if every matrix  $A$  over  $R$  admits a *diagonal reduction*, i.e., if we can find invertible matrices  $P, Q$  and elements  $d_1, \dots, d_n \in R$  such that  $PAQ = \text{diag}(d_1, \dots, d_n)$  and  $d_1 \mid d_2 \mid \dots \mid d_n$ . Principal ideal domains (PID) are elementary divisor rings [DF04, Theorem 4]. This classical result extends effortlessly to PIRs thanks to a theorem of Hungerford. Indeed, the class of elementary divisor rings is stable under taking quotients and direct sums. As a PIR is a direct sum of rings, each of which is the homomorphic image of a PID [Hun68, Theorem 1], our claim follows. A Noetherian elementary divisor ring has moreover a unique invariant factor decomposition [Kap49, Theorems 9.1 and 9.3], thus we showed

**Lemma 1.** *Let  $R$  be a PIR. Then the following hold:*

- (i)  *$R$  is an elementary divisor ring.*
- (ii) *Every finitely generated  $R$ -module  $M$  has a unique invariant factor decomposition, i.e., a decomposition of the form  $R/\mathfrak{a}_1 \times R/\mathfrak{a}_2 \times \dots \times R/\mathfrak{a}_n$  with  $R \neq \mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \dots \supset \mathfrak{a}_n$  where the factors  $\mathfrak{a}_i$  are uniquely determined by the latter condition. For such decomposition,  $n$  is the minimal number of generators of  $M$ , that is  $n = \text{rk}_R(M)$ .*

The following result of Whitehead [Wei13, Example I.1.11] will come in handy in the proof of Theorem A. Let  $R$  be a unital commutative ring. For  $u \in R^\times$ , we denote by  $D_i(u) \in \text{GL}_n(R)$  the diagonal matrix which coincides with the identity except possibly for its  $(i, i)$ -entry, which is set to  $u$ . Let  $D_n(R^\times)$  be the subgroup of  $\text{GL}_n(R)$  generated by the matrices  $D_i(u)$ . Whitehead's lemma implies that a matrix  $D \in D_n(R^\times)$  lies in  $\text{SL}_n(R)$  if and only if it lies in  $E_n(R)$ .

Corollary C elaborates on Theorem A by showing that the unit  $\delta$  of the theorem identifies with a natural invariant of  $E_n(R)$ -equivalence, namely the determinant in the largest quotient of  $M$  that is a free module. This invariant extends Diaconis-Graham's invariant defined for finitely generated Abelian groups [DG99]. It can be defined for any commutative unital ring  $R$  and any finitely generated  $R$ -module  $M$ . Consider a generating set  $\mathbf{m} = (m_1, \dots, m_n)$  of  $M$  with minimal cardinality. We say that  $r \in R$  is *involved* in a relation of

$M$  with respect to  $\mathbf{m}$  if there is  $(r_i) \in R^n$  such that  $\sum r_i m_i = 0$  and  $r = r_i$  for some  $i$ . Denote by  $\mathfrak{r}(M)$  the set of elements of  $R$  which are involved in a relation of  $M$  with respect to  $\mathbf{m}$ . Clearly,  $\mathfrak{r}(M)$  is an ideal of  $R$  and it is easily checked that  $\mathfrak{r}(M)$  is independent of  $\mathbf{m}$ . Let  $\overline{\mathbf{m}} = \pi(\mathbf{m})$  be the image of  $\mathbf{m}$  by the natural map  $\pi : M \twoheadrightarrow M/\mathfrak{r}(M)M$  and let  $\mathbf{e}$  be the canonical basis of  $(R/\mathfrak{r}(M))^n$ . Then the map  $\overline{\mathbf{m}} \mapsto \mathbf{e}$  induces an isomorphism  $\varphi_{\mathbf{m}}$  from  $M/\mathfrak{r}(M)M$  onto  $(R/\mathfrak{r}(M))^n$ . For  $\mathbf{m}' \in \text{Um}_n(M)$ , we define  $\det_{\mathbf{m}}(\mathbf{m}')$  as the determinant of  $\varphi_{\mathbf{m}} \circ \pi(\mathbf{m}')$  with respect to  $\mathbf{e}$ . Let  $\varphi \doteq \varphi_{\mathbf{m}} \circ \varphi_{\mathbf{m}'}^{-1}$ . Because the identity  $\det_{\mathbf{m}} = \det(\varphi) \det_{\mathbf{m}'}$  holds, we have shown the following

**Lemma 2.** *Let  $n = \text{rk}_R(M)$  and let  $\mathbf{m}, \mathbf{m}' \in \text{Um}_n(M)$ . There exists  $u \in (R/\mathfrak{r}(M))^\times$  such that  $\det_{\mathbf{m}} = u \det_{\mathbf{m}'}$ .*

It is straightforward to check that  $\det_{\mathbf{m}}$  is an invariant of  $E_n(R)$ -equivalence, i.e.,  $\det_{\mathbf{m}}(\mathbf{m}'E) = \det_{\mathbf{m}}(\mathbf{m}')$  for every  $E \in E_n(R)$ . If  $\mathfrak{r}(M) = R$ , then  $\det_{\mathbf{m}}$  is trivial and hence useless. This doesn't happen when  $R$  is an elementary divisor ring, for  $\mathfrak{r}(M)$  is then the first invariant factor  $\mathfrak{a}_1$  of  $M$ . The identity elements of each factor ring in a decomposition  $M \simeq R/\mathfrak{a}_1 \times \cdots \times R/\mathfrak{a}_n$  where  $n = \text{rk}_R(M)$  form a unimodular row of  $M$ . We refer to this unimodular row as the *unimodular row naturally associated* to the given decomposition.

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## 2. PROOFS

*Proof of Theorem A.* The case  $k = 1$  follows straightforwardly from (1). We assume now that  $k \geq 2$  and reverse the order of the sequence  $(\mathfrak{a}_i)$  for notational convenience, supposing that  $\mathfrak{a}_j \subset \mathfrak{a}_{j+1}$  for every  $j$ . Let  $\mathbf{m} = (m_i) \in \text{Um}_n(M)$  with  $n \geq k$ . We set  $m_{ij} \doteq (m_i)_j$  for every  $1 \leq i \leq n$ ,  $1 \leq j \leq k$  and identify  $\mathbf{m}$  with the matrix  $(m_{ij})$ . Applying (1) to every row  $\mathbf{m}_i \doteq (m_{ij})_{1 \leq j \leq k}$  we obtain a matrix  $E \in E_n(R)$  such that  $\mathbf{m}' = \mathbf{m}E$  is a lower-triangular matrix  $(m'_{ij})$  with  $m'_{11} = 1$ . We claim that  $m'_{jj}$  is a unit of  $R/\mathfrak{a}_j$  for every  $2 \leq j \leq k$ . To see this, we consider the image  $\rho_l$  of  $(m'_{ij})_{1 \leq i, j \leq l}$  via the natural map induced by the projection  $R/\mathfrak{a}_{l-1} \twoheadrightarrow R/\mathfrak{a}_l$  for every  $2 \leq l \leq k$ . Since the rows of  $\rho_l$  generate  $(R/\mathfrak{a}_l)^l$ , the matrix  $\rho_l$  is invertible [Mat89, Theorem 2.4]. Therefore  $\det(\rho_l) \in (R/\mathfrak{a}_l)^\times$  and subsequently  $m'_{ll} \in (R/\mathfrak{a}_l)^\times$ . As a result we readily find  $E' \in E_n(R)$  such that  $\mathbf{m}'' \doteq \mathbf{m}'E' = \text{diag}(1, m''_2, \dots, m''_k)$  with  $m''_j \doteq m'_{jj}$  for  $2 \leq j \leq k$ . If  $k = 2$ , we are done. Otherwise we consider the matrices

$E_j \doteq D_j(1/m_j'')D_k(m_j'')$  for  $2 \leq j \leq k-1$ . Since  $E_j \in E_k(R/\mathfrak{a}_j)$  for every  $j$  by Whitehead's lemma, each matrix  $E_j$  has a lift in  $E_k(R)$  and the product of these lifts is a matrix  $E'' \in E_n(R)$  satisfying  $\mathbf{m}''E'' = \text{diag}(1, \dots, 1, m_k''')$  with  $m_k''' \in (R/\mathfrak{a}_k)^\times$ . If  $n > k$ , we can store  $1 - m_k'''$  in the  $(k, k+1)$ -entry of  $\mathbf{m}''$ , then turn  $m_k'''$  into 1 and eventually cancel the  $(k, k+1)$ -entry with obvious elementary column transformations.  $\square$

*Proof of Corollary C.* Since  $\det$  is invariant under elementary row transformations, the 'only if' part is established. To prove the converse, it suffices to show that there is  $u \in (R/\mathfrak{b})^\times$  such that for every  $\mathbf{m} \in M^k$ ,  $\det(\mathbf{m}) = u\delta(\mathbf{m})$  where  $\delta = \delta(\mathbf{m})$  is the unit given by Theorem A. Let  $\pi, \pi' : M \rightarrow (R/\mathfrak{b})^k$  be the  $R$ -epimorphisms naturally induced by the invariant factor decomposition and the decomposition  $R/\mathfrak{b}_1 \times \dots \times R/\mathfrak{b}_k$  respectively. Our claim certainly holds if  $\pi' = \varphi \circ \pi$  for some automorphism  $\varphi$  of  $(R/\mathfrak{b})^k$ .

To prove the latter fact, we first show that  $\mathfrak{b}$  is the first invariant factor in the decomposition of  $M$  if  $k = \text{rk}_R(M)$  and  $\mathfrak{b} = R$  if  $k > \text{rk}_R(M)$ . As  $R$  is a PIR, we can write  $\mathfrak{b}_i = b_i R$  with  $b_i \in R$  for every  $i$ . Let  $A = \text{diag}(b_1, \dots, b_k)$ , with  $A$  square of order  $k$ . Since  $R$  is an elementary divisor ring, the matrix  $A$  admits a diagonal reduction  $\text{diag}(d_1, \dots, d_k)$ . As the ideal generated by the coefficients of  $A$  is invariant under this reduction, we have  $\mathfrak{b} = d_1 R$ . If  $d_1 \notin R^\times$ , the ideals  $d_i R$  correspond to the invariant factors  $\mathfrak{a}_i$  of  $M$  and hence  $k = \text{rk}_R(M)$ . Otherwise  $\mathfrak{b} = R$  and  $\det$  vanishes on  $\text{Um}_k(M)$  while rows in  $\text{Um}_k(M)$  are all  $E_k(R)$ -equivalent by Theorem A. Therefore we can assume that  $k = \text{rk}_R(M)$  and  $\mathfrak{b} = d_1 R$  is a proper ideal of  $R$ .

Considering the unimodular rows naturally associated to our two decompositions of  $M$  as a direct sum of cyclic factors, we see that the existence of  $\varphi$  is established by Lemma 2. This proves the claim and hence the result.  $\square$

*Proof of Corollary D.* The group  $G$  is a  $\mathbb{Z}$ -module and  $\text{Um}_n(G)$  naturally identifies with  $V_n(G)$ . Considering the matrix counterparts of the transformations  $L_{ij}$  and  $I_i$ , it is easily checked that the Nielsen classes in  $V_n(G)$  coincide with the orbits in  $\text{Um}_n(G)$  of  $\text{GL}_n(\mathbb{Z}) = D_n(\{\pm 1\})E_n(\mathbb{Z})$  where  $D_n(\{\pm 1\})$  is the group of diagonal matrices with diagonal entries in  $\{\pm 1\}$ . The result is then a straightforward consequence of Theorem A and Corollary C.  $\square$

## REFERENCES

- [AJLL14] Adel Alahmadi, S. K. Jain, T. Y. Lam, and A. Leroy. Euclidean pairs and quasi-Euclidean rings. *J. Algebra*, 406:154–170, 2014.
- [Coo76] George E. Cooke. A weakening of the Euclidean property for integral domains and applications to algebraic number theory. I. *J. Reine Angew. Math.*, 282:133–156, 1976.

- [DF04] David S. Dummit and Richard M. Foote. *Abstract algebra*. John Wiley & Sons, Inc., Hoboken, NJ, third edition, 2004.
- [DG99] Persi Diaconis and Ronald Graham. The graph of generating sets of an abelian group. *Colloq. Math.*, 80(1):31–38, 1999.
- [Fas11] Jean Fasel. Some remarks on orbit sets of unimodular rows. *Comment. Math. Helv.*, 86(1):13–39, 2011.
- [Guy16a] L. Guyot. Generators in split extensions of Abelian groups by cyclic groups. Preprint, arXiv:1604.08896 [math.GR], 2016.
- [Guy16b] L. Guyot. Generators of split metabelian groups. In preparation, 2016.
- [Hun68] Thomas W. Hungerford. On the structure of principal ideal rings. *Pacific J. Math.*, 25:543–547, 1968.
- [Kap49] Irving Kaplansky. Elementary divisors and modules. *Trans. Amer. Math. Soc.*, 66:464–491, 1949.
- [LM93] Martin Lustig and Yoav Moriah. Generating systems of groups and Reidemeister-Whitehead torsion. *J. Algebra*, 157(1):170–198, 1993.
- [Mat89] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1989. Translated from the Japanese by M. Reid.
- [MR87] J. C. McConnell and J. C. Robson. *Noncommutative Noetherian rings*. Pure and Applied Mathematics (New York). John Wiley & Sons, Ltd., Chichester, 1987. With the cooperation of L. W. Small, A Wiley-Interscience Publication.
- [NN51] Bernhard H. Neumann and Hanna Neumann. Zwei Klassen charakteristischer Untergruppen und ihre Faktorgruppen. *Math. Nachr.*, 4:106–125, 1951.
- [Oan11] Daniel Oancea. A note on Nielsen equivalence in finitely generated abelian groups. *Bull. Aust. Math. Soc.*, 84(1):127–136, 2011.
- [O’M65] O. T. O’Meara. On the finite generation of linear groups over Hasse domains. *J. Reine Angew. Math.*, 217:79–108, 1965.
- [Rao98] Ravi A. Rao. An abelian group structure on orbits of “unimodular squares” in dimension 3. *J. Algebra*, 210(1):216–224, 1998.
- [vdK83] Wilberd van der Kallen. A group structure on certain orbit sets of unimodular rows. *J. Algebra*, 82(2):363–397, 1983.
- [vdK89] Wilberd van der Kallen. A module structure on certain orbit sets of unimodular rows. *J. Pure Appl. Algebra*, 57(3):281–316, 1989.
- [VS76] L. N. Vaseršteĭn and A. A. Suslin. Serre’s problem on projective modules over polynomial rings, and algebraic  $K$ -theory. *Izv. Akad. Nauk SSSR Ser. Mat.*, 40(5):993–1054, 1199, 1976.
- [Wei13] Charles A. Weibel. *The K-book*, volume 145 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2013. An introduction to algebraic  $K$ -theory.

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